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DYNAMIC PROBLEMS FOR A PLANE AND CYLINDRICAL VISCOELASTIC LAYER PARTIALLY ADHERENT TO A STIFF RING*

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The plane problem is examined of the shear-vibration of an infinite stiff viscoelastic layer covering that adheres partially to an undeformable cover-foundation: rigidly along a strip of width $2a$, and in frictionless contact outside this strip. In addition, an analogous axisymmetric problem is considered for a cylindrical viscoelastic layer. The layer is partially adherent to a ring along one surface: rigidly along a band of width $2a$ and without friction outside this band, and it is rigidly adherent to a ring vibrating in the axial direction along the other surface.

Mixed boundary value problems reduce to the solution of an integral equation of the first kind which reduces, in turn, to an infinite system of linear algebraic equations. Certain results are presented of a numerical solution of the problems posed. Solutions are compared for the visco-elastic and corresponding elastic problems. The efficiency of two methods of solving the integral equation, reduction to an infinite system and approximation of its kernel, is compared for the latter problem.

1. We examine the plane problem of steady vibrations of a viscoelastic layer $0 \leqslant z \leqslant h$, $|x| < \infty$ lying on an undeformable foundation $z = 0$. The layer is rigidly aherent to the foundation along the strip $|x| \leqslant a$ of width $2a$ and makes friction-free contact outside this strip. Along the whole upper boundary $z = h$ the layer is rigidly aherent to an undeformable covering vibrating in a tangential direction (problem A). The boundary conditions of problem A have the form

$$\begin{aligned}
&u_x(x, h, t) = U_0 e^{-i\omega t},\ \ u_z(x, h, t) = 0\\
&u_z(x, 0, t) = 0,\ \ |x| < \infty\\
&u_x(x, 0, t) = 0,\ \ |x| \leqslant a;\ \tau_{xz}(x, 0, t) = 0,\ \ |x| > a
\end{aligned}$$

In addition to problem A, we consider an analogous axisymmetric problem for a viscoelastic cylindrical layer $R_0 \leqslant r \leqslant R_h$, $|x| < \infty$ (the third cylindrical coordinate z is replaced here by x for uniformity in the subsequent calculations). The cylindrical layer is rigidly adherent to a fixed undeformable ring along a strip $|x| \leqslant a$ of width $2a$ at its inner surface $r = R_0$ and abuts it without friction outside the strip. Along the whole external surface $r = R_h$ the cylindrical layer is rigidly adherent to an undeformable ring vibrating in the axial direction (problem B1). The boundary conditions of problem B1 have the form

$$\begin{aligned}
&u_x(R_h, x, t) = U_0 e^{-i\omega t},\ \ u_r(R_h, x, t) = 0\\
&u_r(R_0, x, t) = 0,\ \ |x| < \infty\\
&u_x(R_0, x, t) = 0,\ \ |x| \leqslant a;\ \tau_{rx}(R_0, x, t) = 0,\ \ |x| > a
\end{aligned}$$

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We also consider the problem B2 that differs from problem B1 by the mutual interchange of the boundary conditions on the inner and outer surfaces of the cylindrical layer. The boundary conditions of such a problem have the form

$$u_x(R_0, x, t) = U_0 e^{-i\omega t},\ u_r(R_0, x, t) = 0$$
$$u_r(R_h, x, t) = 0,\ |x| < \infty$$
$$u_x(R_h, x, t) = 0,\ |x| \leqslant a;\ \tau_{rx}(R_h, x, t) = 0,\ |x| > a$$

By a Fourier integral transform in the coordinate x the corresponding elastic mixed boundary value problems can be reduced to an integral equation of the first kind in the amplitude values of the tangential contact stresses $T(x)$ on the adhesion section $|x| \leqslant a$

$$\int_{-a}^{a} k(x-\xi) T(\xi)\, d\xi = 2\pi U_0 \Delta f(\varkappa_2), \quad |x| \leqslant a \tag{1.1}$$

Here

$$k(t) = \int_\Gamma K(u) e^{iut} du, \quad K(u) = \frac{M(u)}{N(u)} \tag{1.2}$$
$$N(u) = \sigma_1 (u^2 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_2 - \sigma_1\sigma_2 \operatorname{sh} \sigma_1 \operatorname{ch} \sigma_2)$$
$$M(u) = u^2\sigma_1\sigma_2(\operatorname{ch}\sigma_1 \operatorname{ch}\sigma_2 - 1) -$$
$$\left(u^4 - u^2\frac{\varkappa_1^2+\varkappa_2^2}{2} + \frac{\varkappa_1^2\varkappa_2^2}{2}\right)\operatorname{sh}\sigma_1 \operatorname{sh}\sigma_2$$
$$f(\varkappa_2) = -\frac{\varkappa_2^2}{\cos\varkappa_2}, \quad \Delta = \frac{G}{2h}, \quad \sigma_j = \sqrt{u^2 - \varkappa_j^2}, \quad j = 1, 2$$
$$\varkappa_2^2 = \frac{\rho\omega^2 h^2}{G}, \quad \varkappa_1^2 = \varkappa_2^2 \frac{1-2\nu}{2(1-\nu)} \quad \text{(problem A)}$$
$$N(u) = \sigma_1(\sigma_1\sigma_2 L_{111}^- L_{210}^+ - u^2 L_{100}^+ L_{211}^-)$$
$$M(u) = u^2\sigma_1\sigma_2(L_{210}^+ L_{101}^+ + L_{110}^- L_{201}^+) -$$
$$u^4 L_{100}^- L_{211}^- - \sigma_1^2\sigma_2^2 L_{200}^- L_{111}^- - 2u^2\beta^{-1}$$
$$L_{jmn}^{\pm} = I_m(\sigma_j) K_n(\sigma_j\beta) \pm I_n(\sigma_j\beta) K_m(\sigma_j)$$
$$f(\varkappa_2) = 2\varkappa_2\pi^{-1}(J_0(\varkappa_2\beta) N_1(\varkappa_2) - J_1(\varkappa_2) N_0(\varkappa_2\beta))^{-1}$$
$$\Delta = \frac{G}{R_0}, \quad \beta = \frac{R_h}{R_0}, \quad \sigma_j = \sqrt{u^2 - \varkappa_j^2}, \quad j = 1, 2$$
$$\varkappa_2^2 = \frac{\rho\omega^2 R_0^2}{G}, \quad \varkappa_1^2 = \varkappa_2^2\frac{1-2\nu}{2(1-\nu)} \quad \text{(problem B1)}$$

The functions $M(u), N(u), f(\varkappa_2)$ of the problem B2 differ from the corresponding functions of problem B1 by the mutual replacement

$$Z_w(\sigma_j) \rightleftarrows Z_w(\sigma_j\beta), \ j = 1, 2;\ w = 0, 1$$

where $Z_w(z)$ denotes any of the Bessel functions $I_w(z), K_v(z), J_w(z)$, and $N_w(z)$.

The ρ in (1.2) is the density, G is the shear modulus, ν is Posson's ratio for the elastic material of the layer, and ω is the angular frequency.

The function $K(u)$ of the elastic problem is even, real on the real axis, and meromorphic in the complex plane. The following representation holds:

$$K(u) = K(0) \prod_{n=1}^{\infty} (u^2 - \zeta_n^2)(u^2 - z_n^2)^{-1} \tag{1.3}$$

where ζ_n, z_n are, respectively, the zeros and poles of the function $K(u)$ from the upper half-plane, whose moduli increase monotonically as the number increases to ensure convergence of the infinite product (1.3). A finite number of zeros and poles can lie on the real axis. The contour of integration Γ is selected in conformity with the radiation conditions as in /1/. We also note that the following estimate holds as $|u| \to \infty$

$$K(u) = \gamma |u|^{-1} + O(|u|^{-3}) \tag{1.4}$$

where A $\gamma = -{}^1/_8\varkappa_2^2(3-4\nu)(1-\nu)^{-1}$ for problem A.

In the case of steady harmonic vibrations

$$\sigma(t) = \sigma_0 e^{-i\omega t}, \quad \varepsilon(t) = \varepsilon_0 e^{-i\omega t} \tag{1.5}$$

the integral equation of the viscoelastic problem can be obtained formally from (1.1) by replacing the elastic moduli by the complex viscoelastic moduli /2/. Without loss of generality, we will limit ourselves henceforth to a three-constant linear deformation law /2/

$$G(t) = G_0 + G_1 e^{-t/t_1} \tag{1.6}$$

Here G_0 is the creep shear modulus, G_1 is the instantaneous shear modulus, and t_1 is the relaxation time. We shall also assume that Poisson's ratio is a constant.

The expression for the real and imaginary parts of the complex modulus $G^*(\omega)$ corresponding to the selected relaxation function (1.6), is obtained by substituting (1.5) and (1.6) into the governing relations of linear viscoelasticity

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\,\varepsilon^{\cdot}(\tau)\,d\tau \tag{1.7}$$

Integrating in (1.7), we arrive at the relationship

$$\sigma_0 e^{-i\omega t} = G^*(\omega)\,\varepsilon_0 e^{-i\omega t}$$
$$\operatorname{Re} G^*(\omega) = G_0 + \frac{p^2}{1+p^2}G_1, \quad \operatorname{Im} G^*(\omega) = -\frac{p}{1+p^2}G_1; \quad p = \omega t_1$$

The components of the complex modulus can be found analogously in the case of other linear deformation laws.

The function $K(u)$ of the viscoelastic problem $(0 < p < \infty)$ is even, and meromorphic in the complex plane. The representation (1.3) and the estimate (1.4) hold for $K(u)$. Meanwhile, this function is complex-valued on the real axis and has real zeros and poles. In this case the contour Γ agrees with the real axis.

2. We now turn to the solution of the integral Eq.(1.1). To solve the elastic and viscoelastic problems we will apply the same method. Following /3/ and taking account of the evenness of the desired solution, we seek it in the form

$$T(x) = B_0 + \sum_{n=1}^{\infty} 2B_n e^{i\zeta_n a} \operatorname{ch} i\zeta_n x \tag{2.1}$$

Here B_0, B_n are constants to be determined, and ζ_n are zeros of the function $K(u)$ from the upper half-plane.

Satisfying (1.1) by direct substitution of the series (2.1) /4/, we arrive at an infinite system of linear algebraic equations to determine the unknowns B_0 and B_n (in matrix form)

$$AB = CB + D \tag{2.2}$$
$$A = \{a_{mn}\} = \{(z_m - \zeta_n)^{-1}\},\ B = \{B_n\},\ C = \{c_{mn}\}$$
$$c_{mn} = -\exp(2ia\zeta_n)(z_m + \zeta_n)^{-1},\quad D = \{d_m\} = \{-B_0/z_m\}$$
$$m = 1, 2, \ldots$$
$$B_0 = U_0\Delta f(\varkappa_2)/K(0) = 2U_0\Delta\varkappa_2/\sin\varkappa_2 \text{ (problem A)}$$
$$B_0 = 2U_0\pi^{-1}\Delta\,[J_0(\varkappa_2)N_0(\varkappa_2\beta) - J_0(\varkappa_2\beta)N_0(\varkappa_2)]^{-1} \text{ (problem B1)}$$

The system of the first kind obtained with a singular matrix must be regularized, i.e., by inverting the singular part resulting in a system of the second kind

$$B = A^{-1}CB + A^{-1}D \tag{2.3}$$

Inversion of the matrix A is examined in /3/ and formulas are presented to find the inverse matrix A^{-1} which will not be written down here. It is also shown there that system (2.3) will be quasicompletely regular.

Displacements of the layer surface outside the rigid adhesion section can be found from the formula

$$u(x, 0, t) = \frac{e^{-i\omega t}}{2\pi\Delta\varkappa_2^2}\left[-U_0 f(\varkappa_2) + \int_{-a}^{a} k(x-\xi)\,T(\xi)\,d\xi\right], \quad |x| > a \tag{2.4}$$

Substituting (2.1) into (2.4) and integrating, we obtain

$$u(x, 0, t) = -\frac{U_0 f(\varkappa_2)\,e^{-i\omega t}}{2\pi\Delta\varkappa_2^2} + \frac{B_0}{\Delta\varkappa_2^2}\sum_{m=1}^{\infty} A_m \exp\left[i(z_m(x-a) - \omega t\right] \tag{2.5}$$
$$A_m = \frac{M(z_m)}{N'(z_m)}\left[-\frac{1}{z_m} + \sum_{n=1}^{\infty}\frac{B_n}{B_0}\left(\frac{\exp(2ia\zeta_n)}{\zeta_n - z_m} - \frac{1}{\zeta_n + z_m}\right)\right]$$

Therefore, the solution of the infinite system (2.3) must be known to calculate the displacements by means of (2.5).

3. As a numerical example, we will consider the solution of the viscoelastic problem A for the following values of the parameters:

$$\frac{\rho\omega^2 h^2}{G_0} = \varkappa_2^0 = 4{,}5, \quad \frac{G_1}{G_0} = 1, \quad \lambda = \frac{a}{h} = 0{,}25, \quad U_0 = 1 \tag{3.1}$$

To calculate the contact stresses by means of (2.1), it is necessary to determine the unknown constants B_0, B_n by solving the system of Eq.(2.3). To do this, it is required primarily to find the zeros and poles of the function $K(u)$ in the upper half-plane. Their location depends substnatially on the value of the parameter $p = \omega t_1$. The trajectories of motion of the first three zeros and poles of the function $K(u)$ are shown in Fig.1 as the parameter p changes from zero (open circles) to infinity (crosses). The arrow indicates the direction in which p increases, the end of the arrow corresponds to the value $p = 0.1$, and the dark point to the value $p = 1$. It is seen that the location of the zeros and poles depends most strongly on p when it is small. The zero and pole that are real for $p = 0$ (elastic problem) are comparatively remote from the real axis even for $p = 0.1$, which results in a decrease in the dimensionless contact stresses $T(x)/B_0$.

Knowing ζ_n and z_n we can find B_0 and B_n from the infinite system of linear algebraic Eq.(2.3). The reduction method which is effective for small values of the geometric parameter λ, can be used for its solution. The method can also be used for large λ by increasing the number of equations.

For a numerical solution of system (2.2), it need not certainly be reduced to the form (2.3), that results in awkward formulas, but can be solved directly in the form (2.2). A numerical experiment showed that the solutions of system (2.3) by iterations and of (2.2) by Gauss's method with sampling of the pirncipal element are practically in agreement. This fact was also noted in /5/.

Graphs of the real part of the dimensionless contact stresses $T^*(x) = \mathrm{Re}[T(x)/B_0]$ are presented in Fig.2 for $p = 0, 0.05$, and 0.5 (curves *1, 2, 3*, respectively). As the parameter p increases form 0 to 0.5 the amplutude $T^*(x)$ decreases radically, changes negligibly for $p \in [0.5, 1]$, and increases somewhat as p increases to ∞. For $p = \infty$ we again arrive at the elastic problem. When carrying out the calculations to construct the graphs in Fig.2, system (2.2) was cut off to 20-th order, which enabled us to find $T^*(x)$ to 0.5% accuracy for $|x| \leqslant 0.8$.

The dependence of $T_0 = |T^*(0)|$ on the dimensionless frequency $\varkappa_2^\circ$ is presented in Fig.3 when $\lambda \ll 1$. Curves *1, 2,3,* correspond to $p = 0.15, 0.5$, and 2. It is seen that as the parameter p increases, the resonance frequencies shift to the right.

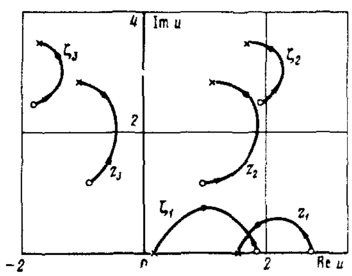

Fig.1

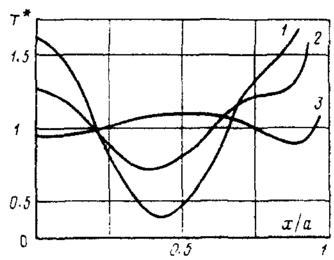

Fig.2

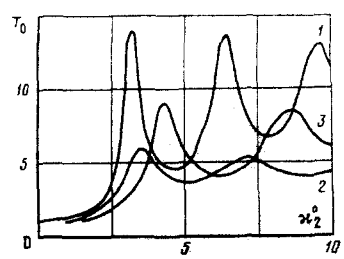

Fig.3

4. Another method of solving the integral Eq.(1.1), based on approximating the function $K(u)$ by another function $K^*(u)$ /3/ whose zeros and poles are easily found, is possible for the elastic problem. The form of the approximating function can be the following, say:

$$K^*(u) = \gamma\varphi(u)\prod_{i=1}^{M}(u^2 - \zeta_i^2)\prod_{i=1}^{L}(u^2 - z_i^2)^{-1}\frac{P_{2N}(u)}{Q_{2N}(u)} \tag{4.1}$$

where $\varphi(u) = u^{-1}\,\mathrm{th}\,\alpha u$ or $\varphi(u) = (u^2 + \alpha^{-2})^{-1/2}$, and α, γ are constants, where the quantity γ is the same as in (1.4), ζ_i, z_i are the real zeros and poles of the function $K(u)$, and P_{2N}, Q_{2N} are reduced polynomials of even equal powers in u whose coefficients are selected from the condition for the best uniform approximation of the function $K(u)$ on the real axis.

Using the arbitrariness in the selection of the parameter α, we set $\alpha \ll 1$. In the case when $\varphi(u) = u^{-1}\,\mathrm{th}\,\alpha u$ the zeros of the function tanh αu by which the solution of (2.1) is constructed, will here be very large compared with the zeros of the polynomial $P_{2N}(u)$, and the components corresponding to the real zeros ζ_n $(n = 1, 2, \ldots, M)$ of the function $K(u)$ and the zeros ζ_n^* $(n = M+1, \ldots, M+N)$ of the polynomial $P_{2N}(u)$ will yield the main contribution to the solution

$$T(x) = B_0 + \sum_{n=1}^{M} 2B_n \exp(i\zeta_n a)\,\mathrm{ch}\,(i\zeta_n x) + \sum_{n=M+1}^{M+N} 2B_n \exp(i\zeta_n^* a)\,\mathrm{ch}\,(i\zeta_n^* x) + O\left(\exp\left(-\frac{\pi a}{\alpha}\right)\right) \tag{4.2}$$

Setting $\alpha \ll 1$ in the approximation (4.1) for $\varphi(u) = (u^2 + \alpha^{-2})^{-1/2}$, the branch point $u = \alpha^{-1}i$ of the function $K^*(u)$ can be raised as high as desired, which will enable the values of the integrals along the slit edges $|u| > |\alpha^{-1}i|$, $\arg u = \pi/2$ to be estimated and made as small as desired, as we do to close the contour Γ in the upper half-plane. In this case, when using the approximation (4.1) for any of the functions $\varphi(u)$ mentioned, truncating the infinite system (2.2) to the $(M+N)$-th order will be natural.

As a numerical example, we will examine the solution of the elastic problem A $(p = 0)$ for values of the parameters (3.1). For $N = 2$ an approximating function of the form (4.1) is

$$K^*(u) = \gamma\varphi(u)\frac{u^2 - \zeta_1^2}{u^2 - z_1^2}\,\frac{u^4 + Au^2 + B}{u^4 + Cu^2 + D} \tag{4.3}$$

$$\varphi(u) = u^{-1}\operatorname{th}\alpha u, \quad \gamma\alpha\frac{\zeta_1^2}{z_1^2}\,\frac{B}{D} = K(0)$$

$$\gamma = -6{,}508929,\ \zeta_1 = 1{,}82985,\ z_1 = 2{,}69083,\ \alpha = 0{,}1\ A = 30{,}3394,\ B = 134{,}45382,$$
$$C = 1{,}29137,\ D = 3{,}87876$$

The error in such an approximation does not exceed 4%. To determine the constants B_n in solution (4.2) when using approximation (4.3), the system truncated to the third order must be solved where $z_{2,3}^* = \pm 1{,}14349 + 0{,}81356i$, $\zeta_2^* = 2{,}32138i$, $\zeta_3^* = 4{,}99506i$.

The dimensionless contact stresses $T^*(x)$ found by the approximation method have a 16% error in amplitude. As the parameter λ decreases, the error increases and for $\lambda = 0{.}15$ reaches 25%. The error was estimated by comparing with the solution found in Sect.3 by the method of reduction (curve *1*, Fig.2), which had comparatively high accuracy.

An analogous pattern is observed even when using an approximation of the form (4.3) with $\varphi(u) = (u^2 + \alpha^{-2})^{-1/2}$, $\alpha^{-2} = 60$, $A = 21{,}03274$, $B = 131{,}23141$, $C = 1{,}3782173$, $D = 4{,}887442$ (γ, ζ_1, z_1 are the same as in (4.3)), whose error is 6%.

Therefore, the method of approximation should be used with care.

To reduce the error, the order of the polynomials $P_{2N}(u)$ and $Q_{2N}(u)$ should be increased, but this complicates the calculation substantially. Meanwhile, if the first complex zeros and poles of the functions $K(u)$ are known from the upper half-plane $\zeta_{2,3} = \pm 1.8603 + 2.4859i$, $z_{2,3} = \pm 0.9875 + 1.3009i$, then A, B, C, D, can be determined without solving the problem of minimizing the functional

$$F(K^*) = \max_{u \in [0,\infty)} |K(u) - K^*(u)|$$

and setting

$$u^4 + Au^2 + B = (u^2 - \zeta_2^2)(u^2 - \zeta_3^2)$$
$$u^4 + Cu^2 + D = (u^2 - z_2^2)(u^2 - z_3^2)$$

The parameter α is not selected arbitrarily here, but by starting from the condition $K^*(0) = K(0)$. In this case we obtain the following values of the parameters therein: $\alpha^{-2} = 14.1973$, $A = 5.43797$, $B = 92.9376$, $D = 7.11554$, $C = 1.43437$ for approximation (4.3) with $\varphi(u) = (u^2 + \alpha^{-2})^{-1/2}$. The error in the approximation on the real axis does not exceed 5.5%. Knowing the subsequent poles and zeros of the function $K(u)$, we can raise the order of the polynomials $P_{2N}(u)$ and $Q_{2N}(u)$ and obtain a corresponding approximation with smaller error. Thus for $N = 4$ the error is 3% ($\alpha^{-2} = 43.6148$), while for $N = 6$ the error is reduced to 2% ($\alpha^{-2} = 87.5302$), etc.

Partial adhesion of the layer to the undeformable cover was assumed above, where the domain of rigid adhesion Ω is the segment $[-a; +a]$. It should be noted that the problems considered can be solved even for domains Ω of the form

$$[-a; -b] \cup [+b; +a] \quad \text{and} \quad (-\infty; -a] \cup [+a; +\infty)$$

by the same methods as in /6/.

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